COMBINATORIAL IDENTITIES INVOLVING THE MÖBIUS FUNCTION

M. EL BACHRAOUI AND M. SALIM

ABSTRACT. In this paper we derive some identities and inequalities on the Möbius mu function. Our main tool is phi functions for intervals of positive integers and their unions.

1. Introduction

The *Möbius mu* function μ is an important arithmetic function in number theory and combinatorics which appears in various identities. We mention the following identities which are well-known and can be found in books on elementary number theory and arithmetic functions. Let n be a positive integer. Then

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1. \end{cases}$$

If $\tau(n)$ is the number of divisors of n, then

$$\sum_{d|n} \mu(d)\tau(n/d) = 1.$$

If $n = \prod_{i=1}^{r} p_i^{k_i}$ is the prime decomposition of n, then

$$\sum_{d|n} \mu(d)\lambda(d) = 2^r,$$

where λ denotes the *Liouville lambda* function defined as follows:
If $m = \prod_{i=1}^{s} p_i^{l_i}$ is the prime decomposition of m, then

$$\lambda(m) = (-1)^{\sum_{i=1}^{s} l_i}.$$

In this note we give some other identities on the Möbius mu function. Our proofs are combinatorial with phi functions as a main tool. For a survey on combinatorial identities we refer to [6, 9] and their references. We now list some examples of identities which we intend to prove. Let m and n be positive integers such that $n > 1$. Then

Date: *October 20, 2009*.
1991 *Mathematics Subject Classification.* 11A25, 11B05, 11B75.
Key words and phrases. Combinatorial identities, Möbius function, Relatively prime sets, Phi functions, Möbius inversion formula.
Corresponding author: M. El Bachraoui, E-mail: melbachraoui@uaeu.ac.ae.

1)

$$\sum_{d|n} \mu(d) 2^{\lfloor m/d \rfloor - \lfloor (m-1)/d \rfloor} = \sum_{d|n} \mu(d) \left(\lfloor m/d \rfloor - \lfloor (m-1)/d \rfloor \right)$$
$$= \begin{cases} 0, \text{ if } \gcd(m,n) > 1 \\ 1, \text{ if } \gcd(m,n) = 1. \end{cases}$$

2)

$$\sum_{d|n} \mu(d) 2^{\lfloor (m+1)/d \rfloor - \lfloor (m-1)/d \rfloor} = \sum_{d|n} \mu(d) \binom{\lfloor (m+1)/d \rfloor - \lfloor (m-1)/d \rfloor + 1}{2}$$
$$= 1 + \sum_{d|n} \mu(d) \lfloor (m+1)/d \rfloor - \lfloor (m-1)/d \rfloor$$
$$= \begin{cases} 1, \text{ if } (m,n) > 1 \text{ and } (m+1,n) > 1 \\ 2, \text{ if } (m,n) = 1 \text{ and } (m+1,n) > 1 \text{ or } (m,n) > 1 \text{ and } (m+1,n) = 1 \\ 3, \text{ if } (m,n) = (m+1,n) = 1. \end{cases}$$

For the sake of completeness we include the following result which is a natural extension of [3, Theorem 2 (a)] on Möbius inversion for arithmetical functions in several variables. For simplicity we let

$$(\overline{m}_a, \overline{n}_b) = (m_1, m_2, \ldots, m_a, n_1, n_2, \ldots, n_b)$$

and

$$\left(\frac{\overline{m}_a}{d}, \left[\frac{\overline{n}_b}{d} \right] \right) = \left(\frac{m_1}{d}, \frac{m_2}{d}, \ldots, \frac{m_a}{d}, \left[\frac{n_1}{d} \right], \left[\frac{n_2}{d} \right], \ldots, \left[\frac{n_b}{d} \right] \right).$$

Theorem 1. *If F and G are arithmetical of $a+b$ variables, then*

$$G(\overline{m}_a, \overline{n}_b) = \sum_{d|(m_1, m_2, \ldots, m_a)} F\left(\frac{\overline{m}_a}{d}, \left[\frac{\overline{n}_b}{d} \right] \right)$$

if and only if

$$F(\overline{m}_a, \overline{n}_b) = \sum_{d|(m_1, m_2, \ldots, m_a)} \mu(d) G\left(\frac{\overline{m}_a}{d}, \left[\frac{\overline{n}_b}{d} \right] \right).$$

2. Phi functions

Throughout let k, l, m, l_1, l_2, m_1, m_2 and n be positive integers such that $l \leq m$, $l_1 \leq m_1$ and $l_2 \leq m_2$, let $[l, m] = \{l, l+1, \ldots, m\}$, and let A be a nonempty finite set of positive integers. The set A is called *relatively prime to* n if $\gcd(A \cup \{n\}) = \gcd(A, n) = 1$.

Definition 2. *Let*

$$\Phi(A, n) = \#\{X \subseteq A : \ X \neq \emptyset \textit{ and } \gcd(X, n) = 1\}$$

and

$$\Phi_k(A, n) = \#\{X \subseteq A : \ \#X = k \textit{ and } \gcd(X, n) = 1\}.$$

Nathanson, among other things, introduced $\Phi(n)$ and $\Phi_k(n)$ (in our terminology $\Phi([1,n],n)$ and $\Phi_k([1,n],n)$ respectively) along with their formulas in [7]. Formulas for $\Phi([m,n],n)$ and $\Phi_k([m,n],n)$ can be found in [3, 8] and formulas for $\Phi([1,m],n)$ and $\Phi_k([1,m],n)$ are obtained in [4]. Ayad and Kihel in [1, 2] considered extensions to sets in arithmetic progression and obtained formulas for $\Phi([l,m],n)$ and $\Phi_k([l,m],n)$ as consequences. Recently the following formulas for $\Phi([1,m_1]\cup[l_2,m_2])$ and $\Phi_k([1,m_1]\cup[l_2,m_2])$ have been found in [5].

Theorem 3. *We have*

$$(a)\quad \Phi([1,m_1]\cup[l_2,m_2],n)=\sum_{d|n}\mu(d)2^{\lfloor\frac{m_1}{d}\rfloor+\lfloor\frac{m_2}{d}\rfloor-\lfloor\frac{l_2-1}{d}\rfloor},$$

$$(b)\quad \Phi_k([1,m_1]\cup[l_2,m_2],n)=\sum_{d|n}\mu(d)\binom{\lfloor\frac{m_1}{d}\rfloor+\lfloor\frac{m_2}{d}\rfloor-\lfloor\frac{l_2-1}{d}\rfloor}{k}.$$

3. Phi functions for $[l_1,m_1]\cup[l_2,m_2]$

We need the following two lemmas.

Lemma 4. *Let*

$$\Psi(l_1,m_1,l_2,m_2,n)=\#\{X\subseteq[l_1,m_1]\cup[l_2,m_2]:\ l_1,l_2\in X\ \textit{and}\ \gcd(X,n)=1\},$$

$$\Psi_k(l_1,m_1,l_2,m_2,n)=\#\{X\subseteq[l_1,m_1]\cup[l_2,m_2]:\ l_1,l_2\in X,\ |X|=k,\ \textit{and}\ \gcd(X,n)=1\}.$$

Then

$$(a)\quad \Psi(l_1,m_1,l_2,m_2,n)=\sum_{d|(l_1,l_2,n)}\mu(d)2^{\lfloor\frac{m_1}{d}\rfloor+\lfloor\frac{m_2}{d}\rfloor-\frac{l_1+l_2}{d}},$$

$$(b)\quad \Psi_k(m_1,l_2,m_2,n)=\sum_{d|(l_1,l_2,n)}\mu(d)\binom{\lfloor\frac{m_1}{d}\rfloor+\lfloor\frac{m_2}{d}\rfloor-\frac{l_1+l_2}{d}}{k-2}.$$

Proof. (a) Assume first that $m_2\le n$. Let $\mathcal{P}(l_1,m_1,l_2,m_2)$ denote the set of subsets of $[l_1,m_1]\cup[l_2,m_2]$ containing l_1 and l_2 and let $\mathcal{P}(l_1,m_1,l_2,m_2,d)$ be the set of subsets X of $[l_1,m_1]\cup[l_2,m_2]$ such that $l_1,l_2\in X$ and $\gcd(X,n)=d$. It is clear that the set $\mathcal{P}(l_1,m_1,l_2,m_2)$ of cardinality $2^{m_1+m_2-l_1-l_2}$ can be partitioned using the equivalence relation of having the same gcd (dividing l_1, l_2 and n). Moreover, the mapping $A\mapsto\frac{1}{d}X$ is a one-to-one correspondence between $\mathcal{P}(l_1,m_1,l_2,m_2,d)$ and the set of subsets Y of $[l_1/d,\lfloor m_1/d\rfloor]\cup[l_2/d,\lfloor m_2/d\rfloor]$ such that $l_1/d,l_2/d\in Y$ and $\gcd(Y,n/d)=1$. Then

$$\#\mathcal{P}(l_1,m_1,l_2,m_2,d)=\Psi(l_1/d,\lfloor m_1/d\rfloor,l_2/d,\lfloor m_2/d\rfloor,n/d).$$

Thus

$$2^{m_1+m_2-l_1-l_2}=\sum_{d|(l_1,l_2,n)}\#\mathcal{P}(l_1,m_1,l_2,m_2,d)=\sum_{d|(l_1,l_2,n)}\Psi(l_1/d,\lfloor m_1/d\rfloor,l_2/d,\lfloor m_2/d\rfloor,n/d),$$

which by Theorem 1 is equivalent to

$$\Psi(l_1,m_1,l_2,m_2,n)=\sum_{d|(l_1,l_2,n)}\mu(d)2^{\lfloor m_1/d\rfloor+\lfloor m_2/d\rfloor-(l_1+l_2)/d}.$$

Assume now that $m_2 > n$ and let a be a positive integer such that $m_2 \leq n^a$. As $\gcd(X,n) = 1$ if and only if $\gcd(X,n^a) = 1$ and $\mu(d) = 0$ whenever d has a nontrivial square factor, we have

$$\begin{aligned}
\Psi(l_1,m_1,l_2,m_2,n) &= \Psi(l_1,m_1,l_2,m_2,n^a) \\
&= \sum_{d|(l_1,l_2,n^a)} \mu(d) 2^{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - (l_1+l_2)/d} \\
&= \sum_{d|(l_1,l_2,n)} \mu(d) 2^{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - (l_1+l_2)/d}.
\end{aligned}$$

(b) For the same reason as before, we may assume that $m_2 \leq n$. Noting that the correspondence $X \mapsto \frac{1}{d}X$ defined above preserves the cardinality and using an argument similar to the one in part (a), we obtain the following identity

$$\binom{m_1+m_2-l_1-l_2}{k-2} = \sum_{d|(l_1,l_2,n)} \Psi_k(l_1/d, \lfloor m_1/d \rfloor, l_2/d, \lfloor m_2/d \rfloor, n/d)$$

which by Theorem 1 is equivalent to

$$\Psi_k(l_1,m_1,l_2,m_2,n) = \sum_{d|(l_1,l_2,n)} \mu(d) \binom{\lfloor m_1/d \rfloor + \lfloor m_2/d \rfloor - (l_1+l_2)/d}{k-2}.$$

□

By arguments similar to the ones in the proof of Lemma 4 we have:

Lemma 5. *Let*

$$\psi(l,m,n) = \#\{X \subseteq [l,m] : \ l \in X \ \textit{and} \ \gcd(X,n) = 1\},$$

$$\psi_k(l,m,n) = \#\{X \subseteq [l,m] : \ l \in X, \ \#X = k, \ \textit{and} \ \gcd(X,n) = 1\}.$$

Then

$$\psi(l,m,n) = \sum_{d|(l,n)} \mu(d) 2^{\lfloor m/d \rfloor - l/d},$$

$$\psi_k(l,m,n) = \sum_{d|(l,n)} \mu(d) \binom{\lfloor m/d \rfloor - l/d}{k}.$$

We are now ready to prove the main theorem of this section.

Theorem 6. *We have*

$$(a) \quad \Phi([l_1,m_1] \cup [l_2,m_2]) = \sum_{d|n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_1-1}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor},$$

$$(b) \quad \Phi_k([l_1,m_1] \cup [l_2,m_2]) = \sum_{d|n} \mu(d) \binom{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_1-1}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor}{k}.$$

Proof. (a) Clearly

$$\Phi([l_1, m_1] \cup [l_2, m_2]) =$$

$$\Phi([1, m_1] \cup [l_2, m_2]) - \sum_{i=1}^{l_1-1} \sum_{j=l_2}^{m_2} \Psi(i, m_1, j, m_2, n) - \sum_{i=1}^{l_1-1} \psi(i, m_1, n) = \tag{1}$$

$$\sum_{d|n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor} - \sum_{i=1}^{l_1-1} \sum_{j=l_2}^{m_2} \sum_{d|(i,j,n)} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{i+j}{d}} - \sum_{i=1}^{l_1-1} \sum_{d|(i,n)} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor - \frac{i}{d}},$$

where the second identity follows by Theorem 3, Lemma 4, and Lemma 5. Rearranging the triple summation in identity (1), we get

$$\begin{aligned}
\sum_{i=1}^{l_1-1} \sum_{j=l_2}^{m_2} \sum_{d|(i,j,n)} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{i+j}{d}} &= \sum_{d|n} \sum_{\substack{i=1 \\ d|i}}^{l_1-1} \sum_{\substack{j=l_2 \\ d|j}}^{m_2} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \frac{i+j}{d}} \\
&= \sum_{d|n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor} \sum_{i=1}^{\lfloor \frac{l_1-1}{d} \rfloor} 2^{-i} \sum_{j=\lfloor \frac{l_2-1}{d} \rfloor + 1}^{\lfloor \frac{m_2}{d} \rfloor} 2^{-j} \\
&= \sum_{d|n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor} (1 - 2^{-\lfloor \frac{m_2}{d} \rfloor + \lfloor \frac{l_2-1}{d} \rfloor}) \sum_{i=1}^{\lfloor \frac{l_1-1}{d} \rfloor} 2^{-i} \qquad (2) \\
&= \sum_{d|n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor} (1 - 2^{-\lfloor \frac{m_2}{d} \rfloor + \lfloor \frac{l_2-1}{d} \rfloor})(1 - 2^{-\lfloor \frac{l_1-1}{d} \rfloor}).
\end{aligned}$$

Similarly the double summation in identity (1) gives

$$\sum_{i=1}^{l_1-1} \sum_{d|(i,n)} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor - \frac{i}{d}} = \sum_{d|n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor} (1 - 2^{-\lfloor \frac{l_1-1}{d} \rfloor}). \tag{3}$$

Combining identities (1), (2), and (3) we find

$$\Phi([l_1, m_1] \cup [l_2, m_2]) = \sum_{d|n} \mu(d) 2^{\lfloor \frac{m_1}{d} \rfloor + \lfloor \frac{m_2}{d} \rfloor - \lfloor \frac{l_1-1}{d} \rfloor - \lfloor \frac{l_2-1}{d} \rfloor}.$$

This completes the proof of part (a). Part (b) follows by similar ideas. □

Corollary 7. ([2]) *We have*

$$\Phi([l, m], n) = \sum_{d|n} 2^{\lfloor \frac{m}{d} \rfloor - \lfloor \frac{l-1}{d} \rfloor},$$

$$\Phi_k([l, m], n) = \sum_{d|n} \binom{\lfloor \frac{m}{d} \rfloor - \lfloor \frac{l-1}{d} \rfloor}{k}.$$

Proof. Use Theorem 6 with $l_1 = l$, $m_1 = m - 1$, and $l_2 = m_2 = m$. □

4. Combinatorial identities

In this section we assume that $n > 1$.

Theorem 8. *We have:*

(a) $$\sum_{d|n} \mu(d) 2^{\lfloor \frac{m}{d} \rfloor - \lfloor \frac{m-1}{d} \rfloor} = \sum_{d|n} \mu(d) \left(\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor \right) = \begin{cases} 0, & \textit{if } \gcd(m,n) > 1 \\ 1, & \textit{if } \gcd(m,n) = 1. \end{cases}$$

(b) $$\sum_{d|n} \mu(d) 2^{\lfloor \frac{m+1}{d} \rfloor - \lfloor \frac{m-1}{d} \rfloor} =$$

$$\begin{cases} 1, & \textit{if } (m,n) > 1 \textit{ and } (m+1,n) > 1 \\ 2, & \textit{if } (m,n) = 1 \textit{ and } (m+1,n) > 1 \textit{ or } (m,n) > 1 \textit{ and } (m+1,n) = 1 \\ 3, & \textit{if } (m,n) = (m+1,n) = 1. \end{cases}$$

Proof. (a) Apply Corollary 7 to $l = m$ and $k = 1$.
(b) Apply Corollary 7 to the interval $[m, m+1]$. □

Theorem 9. *We have:*

(a) $$\sum_{d|n} \mu(d) \left(\left\lfloor \frac{m+1}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor \right) =$$

$$\begin{cases} 0, & \textit{if } (m,n) > 1 \textit{ and } (m+1,n) > 1 \\ 1, & \textit{if } (m,n) = 1 \textit{ and } (m+1,n) > 1 \textit{ or } (m,n) > 1 \textit{ and } (m+1,n) = 1 \\ 2, & \textit{if } (m,n) = (m+1,n) = 1. \end{cases}$$

(b) $$\sum_{d|n} \mu(d) \binom{\lfloor \frac{m+1}{d} \rfloor - \lfloor \frac{m-1}{d} \rfloor + 1}{2} =$$

$$\begin{cases} 1, & \textit{if } (m,n) > 1 \textit{ and } (m+1,n) > 1 \\ 2, & \textit{if } (m,n) = 1 \textit{ and } (m+1,n) > 1 \textit{ or } (m,n) > 1 \textit{ and } (m+1,n) = 1 \\ 3, & \textit{if } (m,n) = (m+1,n) = 1. \end{cases}$$

Proof. (a) Apply Theorem 6(b) to $l_1 = m$, $m_1 = m+1$, $l_2 = m_2 = n$, and $k = 1$ and use the fact that $\sum_{d|n} \mu(d) = 0$ whenever $n > 1$.
(b) Apply Theorem 6(b) to $l_1 = m$, $m_1 = m+1$, $l_2 = m_2 = n$, and $k = 2$. □

References

[1] Mohamed Ayad and Omar Kihel, *On the Number of Subsets Relatively Prime to an Integer*, Journal of Integer Sequences, **Vol. 11**, (2008), Article 08.5.5.

[2] Mohamed Ayad and Omar Kihel, *On Relatively Prime Sets*, Integers **9**, (2009), 343-352.

[3] Mohamed El Bachraoui, *The number of relatively prime subsets and phi functions for sets* $\{m, m+1, \dots, n\}$, Integers **7** (2007), A43, 8pp.

[4] Mohamed El Bachraoui, *On the Number of Subsets of* $[1, m]$ *Relatively Prime to* n *and Asymptotic Estimates*, Integers **8** (2008), A 41, 5 pp.

[5] Mohamed El Bachraoui, *On Relatively Prime Subsets and Supersets*, Submitted.

[6] M. Hall, *Combinatorial Theory*, Second edition, Wiley, New York, 1986.

[7] Melvyn B. Nathanson, *Affine invariants, relatively prime sets, and a phi function for subsets of* $\{1, 2, \dots, n\}$, Integers **7** (2007), A1, 7pp.

[8] Melvyn B. Nathanson and Brooke Orosz, *Asymptotic estimates for phi functions for subsets of* $\{m+1, m+2, \dots, n\}$, Integers **7** (2007), A54, 5pp.

[9] J. Riordan, *Combinatorial Identities*, Wiley, New York, 1968.

Dept. Math. Sci, United Arab Emirates University, PO Box 17551, Al-Ain, UAE
E-mail address: melbachraoui@uaeu.ac.ae
E-mail address: msalim@uaeu.ac.ae